

# The $\ell^2$ -homology of even Coxeter groups

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## Abstract

Given a Coxeter system  $(W, S)$ , there is an associated CW-complex, denoted  $\Sigma(W, S)$  (or simply  $\Sigma$ ), on which  $W$  acts properly and cocompactly. This is the Davis complex.  $L$ , the nerve of  $(W, S)$ , is a finite simplicial complex. We prove that when  $(W, S)$  is an *even* Coxeter system and  $L$  is a flag triangulation of  $\mathbb{S}^3$ , then the reduced  $\ell^2$ -homology of  $\Sigma$  vanishes in all but the middle dimension. In so doing, our main effort will be examining a certain subspace of  $\Sigma$  called the  $(S, t)$ -ruin, for some  $t \in S$ . To calculate the  $\ell^2$ -homology of this ruin, we subdivide a component of this ruin into subcomplexes we call *colors* and then employ a series of Mayer-Vietoris arguments, taking the union of these colors. Once we have established the  $\ell^2$ -homology of the  $(S, t)$ -ruin, we will be able to calculate that of  $\Sigma$ .

## 1 Introduction

The following conjecture is attributed to Singer.

**Singer's Conjecture 1.1.** *If  $\widetilde{M}^n$  is a closed aspherical manifold, then the reduced  $\ell^2$ -homology of  $\widetilde{M}^n$ ,  $\mathcal{H}_*(\widetilde{M}^n)$ , vanishes for all  $*$   $\neq \frac{n}{2}$ .*

Singer's conjecture holds for elementary reasons in dimensions  $\leq 2$ . Indeed, top-dimensional cycles on manifolds are constant on each component, so a square-summable cycle on an infinite component is constant 0. As a result, Conjecture 1.1 in dimension  $\leq 2$  follows from Poincaré duality. In [9], Lott and Lück proved that it holds for those aspherical 3-manifolds for which Thurston's Geometrization Conjecture is true. (Hence, by Pardon, all aspherical 3-manifolds.) For details on  $\ell^2$ -homology theory, see [6], [7] and [8].

Let  $S$  be a finite set of generators. A *Coxeter matrix* on  $S$  is a symmetric  $S \times S$  matrix  $M = (m_{st})$  with entries in  $\mathbb{N} \cup \{\infty\}$  such that each diagonal entry is 1 and each off diagonal entry is  $\geq 2$ . The matrix  $M$  gives a presentation for an associated *Coxeter group*  $W$ :

$$W = \langle S \mid (st)^{m_{st}} = 1, \text{ for each pair } (s, t) \text{ with } m_{st} \neq \infty \rangle. \quad (1.1)$$

The pair  $(W, S)$  is called a *Coxeter system*. Denote by  $L$  the nerve of  $(W, S)$ . In several papers (e.g., [3], [4], and [6]), M. Davis describes a construction which

associates to any Coxeter system  $(W, S)$ , a simplicial complex  $\Sigma(W, S)$ , or simply  $\Sigma$  when the Coxeter system is clear, on which  $W$  acts properly and cocompactly. The two salient features of  $\Sigma$  are that (1) it is contractible and (2) it admits a cellulation under which the nerve of each vertex is  $L$ . It follows that if  $L$  is a triangulation of  $\mathbb{S}^{n-1}$ ,  $\Sigma$  is an  $n$ -manifold. There is a special case of Singer's conjecture for such manifolds.

**Singer's Conjecture for Coxeter groups 1.2.** *Let  $(W, S)$  be a Coxeter system such that its nerve,  $L$ , is a (weighted) triangulation of  $\mathbb{S}^{n-1}$ . Then*

$$\mathcal{H}_i(\Sigma(W, S)) = 0 \text{ for all } i \neq \frac{n}{2}.$$

In [7], Davis and Okun prove that if Conjecture 1.2 for *right-angled* Coxeter systems is true in some odd dimension  $n$ , then it is also true for right-angled systems in dimension  $n+1$ . (A Coxeter system is right-angled if generators either commute or have no relation.) They also show that Thurston's Geometrization Conjecture holds for these Davis 3-manifolds arising from right-angled Coxeter systems. Hence, the Lott and Lück result implies that Conjecture 1.2 for right-angled Coxeter systems is true for  $n = 3$  and, therefore, also for  $n = 4$ . (Davis and Okun also show that Andreev's theorem, [1, Theorem 2], implies Conjecture 1.2 in dimension 3 for right-angled systems. In fact, using methods similar to those in [7], one can show that Andreev's theorem implies 1.2 for arbitrary Coxeter systems.)

Right-angled Coxeter systems are specific examples of *even* Coxeter systems. We say a Coxeter system is even if for any two generators  $s \neq t$ ,  $(st)^{m_{st}} = 1$  implies that  $m_{st}$  is even. The purpose of this paper is to prove the following:

**The Main Theorem 1.3.** *Let  $(W, S)$  be an even Coxeter system whose nerve  $L$  is a flag triangulation of  $\mathbb{S}^3$ . Then  $\mathcal{H}_i(\Sigma(W, S)) = 0$  for  $i \neq 2$ .*

In order to prove Theorem 1.3, we define a certain subspace  $\Omega$  of  $\Sigma$ , and its boundary  $\partial\Omega$ . We call the pair  $(\Omega, \partial\Omega)$  a *ruin*. We then subdivide  $\Omega$  into subspaces we call "boundary collars," which are isomorphic to  $B \times [0, 1]$ , where  $B$  is a component of  $\partial\Omega$ . We paint these boundary collars finitely many *colors*, which can be categorized as even or odd. The painting of  $\Omega$  is virtually invariant under the group action on  $\Omega$ . Moreover, the intersection of two even colors is 2-acyclic and the intersection of an odd color with all the evens is acyclic. Then using Mayer-Vietoris, we are able to prove that  $\mathcal{H}_*(\Omega, \partial\Omega) = 0$  for  $* = 3, 4$ .

Next, we prove that for any  $V \subseteq S$ , and any  $t \in V$ ,  $\mathcal{H}_*(\Sigma(W_V, V)) \cong \mathcal{H}_*(\Sigma(W_{V-t}, V-t))$ , where  $W_V$  is the subgroup of  $W$  generated by the elements of  $V$ . It follows from induction and Poincaré duality that 1.3 is true.

## 2 Coxeter systems and the complex $\Sigma$

### Coxeter systems.

Given a subset  $U$  of  $S$ , define  $W_U$  to be the subgroup of  $W$  generated by the elements of  $U$ .  $(W_U, U)$  is a Coxeter system. A subset  $T$  of  $S$  is *spherical* if

$W_T$  is a finite subgroup of  $W$ . In this case, we will also say that the subgroup  $W_T$  is spherical. We say the Coxeter system  $(W, S)$  is *even* if for any  $s, t \in S$  with  $s \neq t$ ,  $m_{st}$  is either even or infinite.

Given  $w \in W$ , we call an expression  $w = s_1 s_2 \cdots s_n$  *reduced* if there does not exist an integer  $m < n$  with  $w = s'_1 s'_2 \cdots s'_m$ . Define the *length of  $w$* , or  $l(w)$ , to be the integer  $n$  such that  $s_1 s_2 \cdots s_n$ ,  $s_i \in S$ , is a reduced expression for  $w$ . Denote by  $S(w)$  the set of elements of  $S$  which comprise a reduced expression for  $w$ . This set is well-defined, [4, Proposition 4.1.1].

For  $T \subseteq S$  and  $w \in W$ , the coset  $wW_T$  contains a unique element of minimal length. This element is said to be  $(\emptyset, T)$ -reduced. Moreover, it is shown in [2, Ex. 3, pp. 31-32], that an element is  $(\emptyset, T)$ -reduced if and only if  $l(wt) > l(w)$  for all  $t \in T$ . Likewise, we can define the  $(T, \emptyset)$ -reduced elements to be those  $w$  such that  $l(tw) > l(w)$  for all  $t \in T$ . So given  $X, Y \subseteq S$ , we say an element  $w \in W$  is  $(X, Y)$ -reduced if it is both  $(X, \emptyset)$ -reduced and  $(\emptyset, Y)$ -reduced.

**Shortening elements of  $W$ .** We have the so-called “Exchange” (**E**) condition for Coxeter systems ([2, Ch 4. Section 1, Lemma 3] or [4, Theorem 3.3.4]):

- (**E**) Given a reduced expression  $w = (s_1 \cdots s_k)$  and an element  $s \in S$ , either  $\ell(sw) = k + 1$  or there is an index  $i$  such that

$$sw = (s_1 \cdots \widehat{s_i} \cdots s_k).$$

In the case of even Coxeter systems, the parity of a given generator in the set expressions for an element of  $W$  is well-defined. (We prove this herein, Lemma 3.4.) So, in (**E**),  $s_i = s$ ; i.e, if an element of  $s \in S$  shortens a given element of  $W$ , it does so by deleting an instance of  $s$  in an expression for  $w$ .

It is also a fact about Coxeter groups ([4, Theorem 3.4.2]) that if two reduced expressions represent the same element, then one can be transformed into the other by replacing alternating subwords of the form  $(sts \dots)$  of length  $m_{st}$  by the alternating word  $(tst \dots)$  of length  $m_{st}$ . The proof of the first of the following two lemmas follows immediately from this.

**Lemma 2.1.** *Let  $t \in S$ ,  $w \in W_{S-t}$  and  $v \in W$  with  $wtv$  reduced. If there exists an  $r \in S(w) - S(v)$  with  $(rt)^2 \neq 1$ , then all  $r$ 's appears to the left of  $t$  in any reduced expression for  $wtv$ .*

**Lemma 2.2.** *Let  $(W, S)$  be an even Coxeter system, let  $t, s \in S$  be such that  $2 < m_{st} < \infty$  and let  $U_{st} = \{r \in S \mid m_{rt} = m_{rs} = 2\}$ . Suppose that  $tstw' = wtv$  (reduced) where  $w' \in W$ ,  $w \in W_{S-t}$  and  $S(v) \subset U_{st} \cup \{s, t\}$ . Then  $S(w) \subseteq U_{st} \cup \{s\}$ .*

*Proof.* Suppose that  $w$  is a counterexample of minimum length.  $w$  cannot start with an element of  $U_{st}$ , since if it did, multiplication on the left by this element would produce a shorter counterexample. Nor can  $w$  begin with  $s$ , since by the exchange condition, multiplication on the left by  $s$  would cancel an  $s$  in  $w'$ , producing a shorter counterexample. Therefore,  $w$  must start with some  $r$  which

either does not commute with  $t$  or does not commute with  $s$ . By minimality we may also assume that every element appearing after  $r$  in  $w$  is from  $U_{st} \cup \{s\}$ .

If  $r$  does not commute with  $t$ , then by 2.1,  $r$  appears to the left of  $t$  in any reduced expression for  $wtv$ ; a contradiction to  $tstw' = wtv$ . If  $r$  does commute with  $t$  but does not commute with  $s$ , then multiply both sides of  $tstw' = wtv$  by  $t$  leaving  $stw' = w''sv'$  (reduced) where  $w''$  begins with  $r$ ,  $S(v') \in U_{st} \cup \{s, t\}$  and  $s \notin S(w'')$ . Then, with  $t$  in 2.1 replaced by  $s$ , we have that  $r$  appears to the left of  $s$  in any reduced expression for  $wtv$ ; a contradiction to  $stw' = w''sv'$ .  $\square$

### The complex $\Sigma$ .

Let  $(W, S)$  be an arbitrary Coxeter system. Denote by  $\mathcal{S}$  the poset of spherical subsets of  $S$ , partially ordered by inclusion. Given a subset  $V$  of  $S$ , let  $\mathcal{S}_{<V} := \{T \in \mathcal{S} \mid T \subset V\}$ . Similar definitions exist for  $>, \leq, \geq$ . For any  $w \in W$  and  $T \in \mathcal{S}$ , we call the coset  $wW_T$  a *spherical coset*. The poset of all spherical cosets we will denote by  $W\mathcal{S}$ .

The poset  $\mathcal{S}_{>\emptyset}$  is an abstract simplicial complex, denote it by  $L$ , and call it the *nerve* of  $(W, S)$ . The vertex set of  $L$  is  $S$  and a non-empty subset of vertices  $T$  spans a simplex of  $L$  if and only if  $T$  is spherical.

Let  $K = |\mathcal{S}|$ , the geometric realization of the poset  $\mathcal{S}$ . It is the cone on the barycentric subdivision of  $L$ , the cone point corresponding to the empty set, thus a finite simplicial complex. Denote by  $\Sigma(W, S)$ , or simply  $\Sigma$  when the system is clear, the geometric realization of the poset  $W\mathcal{S}$ . This is the Davis complex. The natural action of  $W$  on  $W\mathcal{S}$  induces a simplicial action of  $W$  on  $\Sigma$  which is proper and cocompact.  $K$  includes naturally into  $\Sigma$  via the map induced by  $T \rightarrow W_T$ , so we view  $K$  as a subcomplex of  $\Sigma$  and note that it is a strict fundamental domain for the action of  $W$  on  $\Sigma$ .

For any element  $w \in W$ , write  $wK$  for the  $w$ -translate of  $K$  in  $\Sigma$ . Let  $w, w' \in W$  and consider  $wK \cap w'K$ . This intersection is non-empty if and only if  $V = S(w^{-1}w')$  is a spherical subset. In fact,  $wK \cap w'K$  is simplicially isomorphic to  $|\mathcal{S}_{[V, T]}|$ , the geometric realization of  $\mathcal{S}_{[V, T]} := \{V' \in \mathcal{S} \mid V \subseteq V' \subseteq T\}$ .

**A cubical structure on  $\Sigma$ .** For each  $w \in W$ ,  $T \in \mathcal{S}$ , denote by  $w\mathcal{S}_{\leq T}$  the subposet  $\{wW_V \mid V \subseteq T\}$  of  $W\mathcal{S}$ . Put  $n = \text{Card}(T)$ .  $|w\mathcal{S}_{\leq T}|$  has the combinatorial structure of a subdivision of an  $n$ -cube. We identify the sub-simplicial complex  $|w\mathcal{S}_{\leq T}|$  of  $\Sigma$  with this coarser cubical structure and call it a *cube of type  $T$* . Note that the vertices of these cubes correspond to spherical subsets  $V \in \mathcal{S}_{\leq T}$ . (For details on this cubical structure, see [10].)

**A cellulation of  $\Sigma$  by Coxeter cells.**  $\Sigma$  has a coarser cell structure: its cellulation by “Coxeter cells.” (For reference, see [4], [7], and [5].) Suppose that  $T \in \mathcal{S}$ ; then by definition  $W_T$  is finite. Take the canonical representation of  $W_T$  on  $\mathbb{R}^{\text{Card}(T)}$  and choose a point  $x$  in the interior of a fundamental chamber. The *Coxeter cell of type  $T$*  is defined as the convex hull  $C$ , in  $\mathbb{R}^{\text{Card}(T)}$ , of  $W_T x$  (a generic  $W_T$ -orbit). The vertices of  $C$  are in 1-1 correspondence with the elements of  $W_T$ . Furthermore, a subset of these vertices is the vertex set of a face of  $C$  if and only if it corresponds to the set of elements in a coset of the form  $wW_V$ , where  $w \in W_T$  and  $V \subset T$ . Hence, the poset of non-empty faces of  $C$  is naturally identified with the poset  $W_T\mathcal{S}_{\leq T} := \{wW_V \mid w \in W_T, V \subset T\}$ .

Therefore, we can identify the simplicial complex  $\Sigma(W_T, T)$  with the barycentric subdivision of the Coxeter cell of type  $T$ .

Now, for each  $T \in \mathcal{S}^{(k)}$  and  $w \in W$ , the poset  $W\mathcal{S}_{\leq wW_T}$  is isomorphic to the poset  $W_T\mathcal{S}_{\leq T}$  via the map  $vW_V \rightarrow w^{-1}vW_V$ . Thus, the subcomplex of  $\Sigma(W, S)$  which is obtained from the poset  $W\mathcal{S}_{\leq wW_T}$  may be identified with the barycentric subdivision of the  $k$ -cell of type  $T$ . In this way, we put a cell structure on  $\Sigma$  which is coarser than the simplicial structure by identifying each simplicial subcomplex  $|W\mathcal{S}_{\leq wW_T}|$  with a cell of type  $T$ .

We will write  $\Sigma_{cc}$ , when necessary, to denote the Davis complex equipped with this cellulation by Coxeter cells. Under this cellulation, the 0-cells of  $\Sigma_{cc}$  correspond to cosets of  $W_\emptyset$ , i.e. to elements from  $W$ ; and 1-cells correspond to cosets of  $W_s$ ,  $s \in S$ . The features of this cellulation are summarized by the following, from [4].

**Proposition 2.3.** *There is a natural cell structure on  $\Sigma$  so that*

- *its vertex set is  $W$ , its 1-skeleton is the Cayley graph of  $(W, S)$  and its 2-skeleton is a Cayley 2-complex.*
- *each cell is a Coxeter cell.*
- *the link of each vertex is isomorphic to  $L$  (the nerve of  $(W, S)$ ) and so if  $L$  is a triangulation of  $\mathbb{S}^{n-1}$ ,  $\Sigma$  is a topological  $n$ -manifold.*
- *a subset of  $W$  is the vertex set of a cell if and only if it is a spherical coset and*
- *the poset of cells is  $W\mathcal{S}$ .*

### Ruins.

The following subspaces are defined in [5]. Let  $(W, S)$  be a Coxeter system. For any  $U \subseteq S$ , let  $\mathcal{S}(U) = \{T \in \mathcal{S} \mid T \subset U\}$  and let  $\Sigma(U)$  be the subcomplex of  $\Sigma_{cc}$  consisting of all cells of type  $T$ , with  $T \in \mathcal{S}(U)$ .

Given  $T \in \mathcal{S}(U)$ , define three subcomplexes of  $\Sigma(U)$ :

$\Omega(U, T)$  : the union of closed cells of type  $T'$ , with  $T' \in \mathcal{S}(U)_{\geq T}$ ,

$\widehat{\Omega}(U, T)$  : the union of closed cells of type  $T''$ ,  $T'' \in \mathcal{S}(U)$ ,  $T'' \notin \mathcal{S}(U)_{\geq T}$ ,

$\partial\Omega(U, T)$  : the cells of  $\Omega(U, T)$  of type  $T''$ , with  $T'' \notin \mathcal{S}(U)_{\geq T}$ .

The pair  $(\Omega(U, T), \partial\Omega(U, T))$  is called the  $(U, T)$ -ruin. For  $T = \emptyset$ , we have  $\Omega(U, \emptyset) = \Sigma(U)$  and  $\partial\Omega(U, \emptyset) = \emptyset$ .

**The subspace  $\Omega$ .** Let  $t \in S$ . We call the  $(S, t)$ -ruin a *one-letter ruin*. Put  $U := \{s \in S \mid m_{st} < \infty\}$ . The path components of  $\Omega(S, t)$  are indexed by the cosets  $W/W_U$ . Denote by  $\Omega$  the path-component of  $\Omega(S, t)$  with vertex set corresponding  $W_U$ . The action of  $W_U$  on  $\Sigma$  restricts to an action on  $\Omega$ . Let  $\partial\Omega := \Omega \cap \partial\Omega(S, t)$  and put  $K(U) := K \cap \Omega$ . Note that the  $W_U$ -translates of  $K(U)$  cover  $\Omega$ , i.e.  $\Omega = \bigcup_{w \in W_U} wK(U)$ .

If we restrict our attention to cubes of type  $T$ , where  $T \subseteq T'$  for some  $T' \in \mathcal{S}_{\geq t}$ ,  $\Omega$  is a cubical complex and  $\partial\Omega$  is a subcomplex. Moreover, if  $B$  is a component of  $\partial\Omega$ , the space  $D := B \times [0, 1]$  is isomorphic to the union of the  $w$ -translates of  $K(U)$  where  $w$  is a vertex of  $B$ . We call such subspaces *boundary collars*. It is clear that the collection of boundary collars covers  $\Omega$ . We denote by  $\partial_{in}(D)$  the “1-end” of this product and note that it is comprised of 0-simplices corresponding to elements of  $\mathcal{S}_{\geq t}$ . The boundary collars intersect along these “inner” boundaries.

### 3 The $\ell^2$ -homology of $\Omega(S, t)$

Here and for the remainder of the paper, we require that  $(W, S)$  be an even Coxeter system whose nerve  $L$  is a flag triangulation of  $\mathbb{S}^3$ . Fix  $t \in S$  and let  $U, \Omega$  and  $\partial\Omega$  be defined as in 2.

Any  $s \in U$  has the property that  $m_{st} < \infty$ . Let  $S' := \{s \in U \mid m_{st} > 2\}$ , and assume that  $S'$  is not empty. The group  $W_U$  has the following properties.

**Lemma 3.1.** *Suppose that  $L$  is flag. Then for  $s, s' \in S'$ , either  $s = s'$ , or  $m_{ss'} = \infty$ .*

*Proof.* Suppose that  $s \neq s'$  and that  $m_{ss'} < \infty$ . Then  $\{s, s'\} \in \mathcal{S}$ , and since  $s, s'$  are both in  $U$ , the vertices corresponding to  $s, s'$  and  $t$  are pairwise connected in  $L$ .  $L$  is a flag complex, so this implies that  $\{s, s', t\} \in \mathcal{S}$ . But

$$\frac{1}{m_{ss'}} + \frac{1}{m_{st}} + \frac{1}{m_{ts'}} \leq \frac{1}{m_{ss'}} + \frac{1}{4} + \frac{1}{4} \leq 1.$$

This contradicts  $\{s, s', t\}$  being a spherical subset. So we must have that  $m_{ss'} = \infty$ .  $\square$

**Corollary 3.2.** *Let  $s \in S'$  and let  $T \in \mathcal{S}_{\geq \{s, t\}}$ . Then  $m_{ut} = m_{us} = 2$  for  $u \in T - \{s, t\}$ . In other words, the generators from  $T - \{s, t\}$  commute with both  $s$  and  $t$ .*

Let  $L_{st}$  denote the link in  $L$  of the edge corresponding to the vertices  $s$  and  $t$ . The above Corollary states that the generators corresponding to the vertex set of  $L_{st}$  commute with both  $s$  and  $t$ . Denote this set of generators by  $U_{st}$ .

Of particular interest to us will be elements of  $W_U$  with a reduced expression of the form  $tst \cdots st$  for some  $s \in S'$ . Since  $W$  is even, this expression is unique, and we have the following Lemma.

**Lemma 3.3.** *Let  $s \in S'$  and let  $u \in W_{\{s, t\}}$  be such that  $u = tst \cdots st$ , is a reduced expression beginning and ending with  $t$ . Then  $u$  is  $(U - t, U - t)$ -reduced.*

**Lemma 3.4.** *Let  $V, T \subset S$  and consider the function  $g_{VT} : W_V \rightarrow W_T$  induced by the following rule:  $g_{VT}(s) = s$  if  $s \in V \cap T$  and  $g_{VT}(s) = e$  (the identity element of  $W$ ) for  $s \in V - T$ .  $g_{VT}$  is a homomorphism.*

*Proof.* We show that  $g_{VT}$  respects the relations in  $W_V$ . Let  $s, u \in V$  be such that  $(su)^m = 1$ . Then

$$g_{VT}((su)^m) = \begin{cases} (su)^m & \text{if } s \in T, u \in T \\ s^m & \text{if } s \in T, u \notin T \\ u^m & \text{if } u \in T, s \notin T \\ e & \text{if } s \notin T, u \notin T. \end{cases}$$

In all cases, since  $(W_V, V)$  is even,  $g_T((su)^m) = e$ .  $\square$

Then with  $T \in \mathcal{S}_{\geq t}$  and  $U$  as above, we define an action of  $W_U$  on the set of cosets  $W_T/W_{T-t}$ : For  $w \in W_U$  and  $v \in W_T$ , define

$$w \cdot vW_{T-t} = g_{UT}(w)vW_{T-t}. \quad (3.1)$$

**Coloring boundary collars.**

Set

$$A = \prod_{T \in \mathcal{S}_{\geq t}} W_T/W_{T-t}.$$

We call  $A$  the set of colors, note that it is a finite set. The action defined in (3.1) extends to a diagonal  $W_U$ -action on  $A$ . So for  $w \in W_U$  and  $a \in A$ , write  $w \cdot a$  to denote  $w$  acting on  $a$ . Let  $\bar{e}$  be the element of  $A$  defined by taking the trivial coset  $W_{T-t}$  for each  $T \in \mathcal{S}_{\geq t}$ . Vertices of  $\Omega$  correspond to group elements of  $W_U$ , so we paint the vertices of  $\Omega$  by defining a map  $c : W_U \rightarrow A$  with the rule  $c(w) := w \cdot \bar{e}$ .

**Remark 3.5.** If an element  $w \in W_U$  contains no  $t$ 's in any of its reduced expressions, then  $w$  acts trivially on the element  $\bar{e}$ , i.e.  $w \cdot \bar{e} = \bar{e}$ .

We will paint the space  $wK(U)$  with  $c(w)$ . In this way, all of  $\Omega$  is colored by some element of  $A$ . For vertices  $w$  and  $w'$  of the same component  $B$  of  $\partial\Omega$ ,  $h = w^{-1}w' \in W_{U-t}$ . So  $c(w') = c(wh) = wh \cdot \bar{e} = w \cdot \bar{e} = c(w)$ , and therefore all of  $D = B \times [0, 1]$  is painted with  $c(w)$ . Note that each component of  $\partial\Omega$  is monochromatic while  $\partial_{in}(D)$  is not.

**Lemma 3.6.** *Let  $D = B \times [0, 1]$  and  $D' = B' \times [0, 1]$  be boundary collars where  $B$  and  $B'$  are different components of  $\partial\Omega$ . Suppose that the vertices of  $B$  and  $B'$  have the same color. Then  $D \cap D' = \emptyset$ .*

*Proof.* Suppose, by way of contradiction, that  $D \cap D' \neq \emptyset$ , i.e. there exist vertices  $w \in B$ ,  $w' \in B'$  such that  $c(w) = c(w')$  and  $wK(U) \cap w'K(U) \neq \emptyset$ . Let  $V = S(w^{-1}w')$  and  $v = w^{-1}w'$ . Then  $c(w) = c(w') \Rightarrow w \cdot \bar{e} = wv \cdot \bar{e} \Rightarrow \bar{e} = v \cdot \bar{e}$ . Thus, for any  $T \in \mathcal{S}_{\geq t}$ , we have that

$$v \cdot W_{T-t} = W_{T-t}. \quad (3.2)$$

$V \cup t$  is spherical, and since  $v \in W_V$ , the action of  $v$  on  $W_{V \cup t}/W_{V-t}$  defined in (3.1) is left multiplication by  $v$ . So by (3.2), we have that  $v \in W_{V-t}$ . But this contradicts  $w$  and  $w'$  coming from different components of  $\partial\Omega$ .  $\square$

Then for  $c \in A$ , define the  $c$ -collars,  $F_c$ , to be the disjoint union of the boundary collars  $D = B \times [0, 1]$  where each component  $B$  of  $\partial\Omega$  has the color  $c$ . We refer to these collections as *colors*. The collection of colors is a finite cover of  $\Omega$ .

**Even and odd colors.**

Let  $T = \{t\}$  and consider the homomorphism  $g_{UT} : W_U \rightarrow W_t$  defined in (3.4). Under  $g_{UT}$ , an element  $w \in W_U$  is sent to the identity in  $W_t$  if  $w$  has an even number of  $t$ 's present in some factorization (and therefore, all factorizations) as a product of generators from  $U$  and an element  $w \in W_U$  is sent to  $t \in W_t$  if  $w$  has an odd number of  $t$ 's present in some factorization. Thus, we call a vertex  $w$  *even* if  $g_{UT}(w) = e$ ; *odd* if  $g_{UT}(w) = t$ . If two vertices  $w$  and  $w'$  are such that  $c(w) = c(w')$ , then clearly  $g_{UT}(w) = g_{UT}(w')$ . So we may also classify the colors (both the elements of  $A$  and the collections of boundary collars), as even or odd. We will suppress the subscript  $c$  and say a color  $F$  is even or odd.

Of fundamental importance will be how these colors intersect. By Remark 3.5, we know that in order for the vertices of a Coxeter cell to support two different colors, this cell must be of type  $T \in \mathcal{S}_{\geq t}$ . But, for a cell to support two different *even* vertices,  $v$  and  $v'$ , this cell must be of type  $T \in \mathcal{S}_{\geq \{s, t\}}$  for exactly one  $s \in S'$  (uniqueness is given by Corollary 3.2). Moreover,  $w = v^{-1}v'$  has the properties that  $\{s, t\} \subseteq S(w)$  and that it contains at least two, and an even number of  $t$ 's in any factorization as a product of generators. We call such  $w$  *t-even*.

**Example 3.7.** The following example is representative of our situation. Suppose  $L = \mathbb{S}^1$ , and  $U = \{t, r, s \mid (rt)^2 = 1, (st)^4 = 1\}$ .  $\Omega$  is represented in Figure 1. The black dots represent the vertices of the Coxeter cellulation, with the vertices  $e$  and  $tst$  labeled. The even colors are shaded. Even boundary collars intersect in a 0-simplex corresponding to the spherical subset  $\{s, t\}$ . The intersection of one odd color and all evens is the inner boundary of the odd color.

**The intersection of even colors.** Let  $D_0$  denote the boundary collar containing the vertex  $e$ . Fix  $s \in S'$  and let  $D_2$  denote the boundary collar containing the vertex  $u$ , where  $u \in W_{\{s, t\}}$  is  $t$ -even and has a reduced expression ending in  $t$ . We study  $D_0 \cap D_2$ .

**Lemma 3.8.** *Let  $W' := W_{U_{st}}$  and let  $K' = K(U) \cap uK(U)$ . Denote by  $W'K'$  the orbit of  $K'$  under  $W'$ .  $D_0 \cap D_2 = W'K'$ .*

*Proof.* For any  $w \in W'$ , the vertex  $w$  is in the same component of  $\partial\Omega$  as  $e$ , and therefore  $wK(U) \subset D_0$ .  $wu = uw$ , so  $wu$  is in the same component of  $\partial\Omega$  as  $u$  and  $wuK(U) \subset D_2$ . Thus  $wK' = wK(U) \cap uwK(U) \subset D_0 \cap D_2$ .

Now let  $\sigma$  be a 0-simplex in  $D_0 \cap D_2$ . Then there exist  $w, w' \in W_{U-t}$  such that  $\sigma \in wK(U) \cap uw'K(U)$ , i.e.  $\sigma$  is simultaneously the  $w$ - and  $uw'$ -translate of a 0-simplex  $\sigma'$  in  $K(U)$ . Let  $V$  be the spherical subset to which  $\sigma'$  corresponds and let  $v \in W_V$  be such that  $uw' = wv$ .  $c(e) = c(w)$  and  $c(u) = c(uw')$ ,



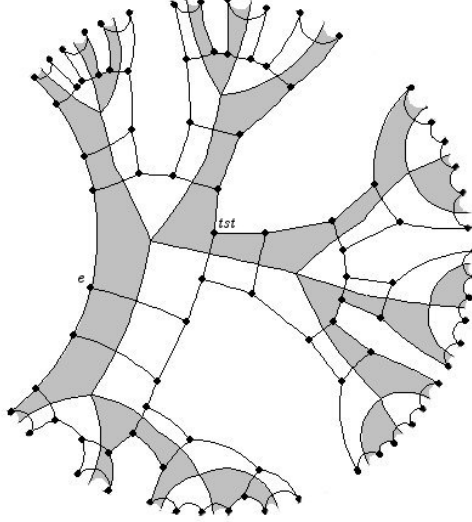


Figure 1: Even and Odd Colors of  $\Omega$

so  $w$  and  $uw'$  are differently colored even vertices of a cell of type  $V$ . Thus  $\{s', t\} \subseteq S(v) \subseteq V$  for exactly one  $s' \in S'$  and  $v$  is  $t$ -even.

**Claim 1:**  $s' = s$ .

**Pf of Claim 1:** Since  $w' \in W_{U-t}$ ,  $c(u) = c(uw') = c(wv)$ , i.e.  $u$  and  $wv$  act the same on every coordinate of  $\bar{e}$ . Consider the  $\{s, t\}$ -coordinate.  $u \in W_{\{s, t\}}$  is  $t$ -even, so  $u \cdot W_s = uW_s$  and  $uW_s \neq W_s$ . But if  $s \notin S(v)$ , then  $v$  being  $t$ -even and  $w \in W_{U-t}$  imply that  $wv \cdot W_s = W_s$ ; which contradicts  $u$  and  $wv$  having the same color. So **Claim 1** is true, and as a result  $V \in \mathcal{S}_{\geq \{s, t\}}$  and  $\sigma' \in K'$ . It remains to show that  $\sigma$  is in the  $W'$ -orbit of  $K'$ .

**Claim 2:**  $S(w) \subseteq (U_{st} \cup \{s\})$ .

**Pf of Claim 2:** Take a reduced expression for  $u$  which ends in  $t$ . If this expression begins with  $s$ , multiply  $u$  on the left by  $s$ , so that we have  $suw' = swv$ . The only change this can effect on  $S(w)$  is by either adding or subtracting an  $s$ , which is inconsequential to our claim. So, we may assume that  $u$  has a reduced expression of the form  $tst \cdots st$  as described in Lemma 3.3. Hence,  $u$  is  $(U - t, U - t)$ -reduced and  $uw'$  has a reduced expression beginning with the subword  $tst$ .  $v$  is  $t$ -even, so  $wv$  has a reduced expression of the form  $wtv'$  where  $w \in W_{U-t}$  and  $S(v') \subset U_{st} \cup \{s, t\}$ . **Claim 2** then follows from Lemma 2.2.

We now finish the proof of 3.8. If  $s \notin S(w)$ , then  $w \in W'$  and we are done since  $\sigma$  is the  $w$ -translate of  $\sigma'$ . If  $s \in S(w)$ , then  $w$  may be written as  $qs$ , with  $q \in W'$  and since  $s \in V$ ,  $qsW_V = qW_V$ . So  $\sigma$  is also the  $q$ -translate of  $\sigma'$ .  $\square$

**Proposition 3.9.**  $(D_0 \cap D_2) \cong \Sigma(W', U_{st})$ , an infinite connected 2-manifold.

*Proof.* Since  $S(u) = \{s, t\}$ ,  $K'$  is the geometric realization of the poset  $\mathcal{S}_{\geq \{s, t\}} = \{V \in \mathcal{S} \mid \{s, t\} \subseteq V\}$ . By Lemma 3.8,  $(D_0 \cap D_2) \cong |W' \mathcal{S}_{\geq \{s, t\}}|$ , and by Corollary

3.2,  $\mathcal{S}_{\geq\{s,t\}}$  is isomorphic to  $\mathcal{S}(U_{st})$  via the map  $T \rightarrow T - \{s, t\}$ . So  $(D_0 \cap D_2) \cong |W'\mathcal{S}(U_{st})| = \Sigma(W', U_{st})$ .

Simplices in  $L_{st}$  correspond to spherical subsets  $T \in \mathcal{S}$  such that neither  $s$  nor  $t$  is contained in  $T$  but  $T \cup \{s, t\} \in \mathcal{S}$ . So by Corollary 3.2, the vertex set of a simplex of  $L_{st}$  corresponds to a spherical subset of  $\mathcal{S}(U_{st})$ . Conversely, given a spherical subset  $T \in \mathcal{S}(U_{st})$ ,  $W_{T \cup \{s, t\}} = W_T \times W_{\{s, t\}}$ , which is finite. So  $T$  corresponds to a simplex of  $L_{st}$ . Thus,  $L_{st}$  is the nerve of the system  $(W', U_{st})$ . Since  $L$  triangulates  $\mathbb{S}^3$ ,  $L_{st}$  triangulates  $\mathbb{S}^1$ . The result follows from Proposition 2.3.  $\square$

**Corollary 3.10.** *Let  $F \neq F'$  be even colors. Then  $\mathcal{H}_2(F \cap F') = 0$ .*

*Proof.* Suppose that  $F \neq F'$  are both even colors such that  $F \cap F' \neq \emptyset$ . Then there exist even vertices  $v$  and  $v'$  with  $vK(U) \cap v'K(U) \neq \emptyset$ . Let  $w = v^{-1}v'$  and put  $T = S(v^{-1}v')$ .  $T$  is a spherical subset, and  $v$  and  $v'$  are both vertices of a cell of type  $T$ . So we have exactly one  $s \in S'$  with  $\{s, t\} \subset T$ . Factor  $w$  as  $w = xq$  where  $x \in W_{\{s, t\}}$  is  $t$ -even and  $q \in W_{T - \{s, t\}}$ . Now,  $x$  may not have a reduced expression ending in  $t$ . If it does not, then  $xs$  does and it is in the same boundary collar as  $x$  and  $w$ . So let

$$u = \begin{cases} x & \text{if } x \text{ has a reduced expression ending in } t, \\ xs & \text{otherwise.} \end{cases}$$

Then  $vK(U) \cap v'(U) \subseteq vK(U) \cap vuK(U)$ . Act on the left by  $v^{-1}$  and we are in the situation studied in 3.8 and 3.9. So  $F \cap F'$  is the disjoint union of infinite connected 2-manifolds. As a result, any 2-cycle must be constant 0.  $\square$

**Remark 3.11.** If  $S' = \{s \in S \mid 2 < m_{st} < \infty\} = \emptyset$ , then  $W_U = W_{U-t} \times W_t$  and there is one even color and one odd color.

**Multiple even colors.** Suppose that  $D_1, D_2, \dots, D_n, D_e$  are even boundary collars. Then

$$D_e \cap \left( \bigcup_{j=1}^n D_j \right) = (D_e \cap D_1) \cup \dots \cup (D_e \cap D_n),$$

and suppose that for some  $1 \leq i < k \leq n$  we have that  $(D_e \cap D_i) \cup (D_e \cap D_k)$  is not disjoint. Let  $\sigma$  be a 0-simplex contained in  $D_e \cap D_i \cap D_k$  corresponding to a coset of the form  $vW_T$ . Then there exists  $w, w' \in W_T$  such that  $v \in D_e$ ,  $vw \in D_i$ ,  $vw' \in D_k$  and  $\sigma \in vK(U) \cap vwK(U) \cap vw'K(U)$ . These three vertices are differently colored even vertices of a cell of type  $T$ , so  $\{s, t\} \subseteq T$  for exactly one  $s \in S'$  and both  $w$  and  $w'$  are  $t$ -even. Then, as in the proof of 3.10, it follows that  $D_e \cap D_i = D_e \cap D_k \cong |W'\mathcal{S}_{\geq\{s, t\}}|$ . So Corollary 3.10 generalizes to the following:

**Corollary 3.12.** *Let  $F_1, F_2, \dots, F_n, F_e$  be even colors. Then*

$$\mathcal{H}_2(F_e \cap \left( \bigcup_{j=1}^n F_j \right)) = 0.$$

**Lemma 3.13.** *Let  $\mathcal{F}_E$  denote the union of all even colors and let  $F_o$  be an odd color. Define*

$$\partial_{in}(F_c) := \coprod_{D \subset F_c} \partial_{in}(D).$$

$$F_o \cap \mathcal{F}_E = \partial_{in}(F_o).$$

*Proof.* Since  $F_o$  is a disjoint union of boundary collars, it suffices to show that  $D \cap \mathcal{F}_E = \partial_{in}(D)$  for some boundary collar  $D \subset F_o$ .

( $\supseteq$ ): Let  $\sigma$  be a 0-simplex in  $\partial_{in}(D)$ . Then  $\sigma$  corresponds to a coset of the form  $wW_V$  where  $V \in \mathcal{S}_{\geq t}$  and  $w \in W_U$  is an odd vertex of  $D$ . Consider the even vertex  $wt$ . Then since  $t \in V$ ,  $wW_V = wtW_V$ , and  $\sigma \in wtK(U) \subset \mathcal{F}_E$ .

( $\subseteq$ ): Now suppose that  $\sigma$  is a 0-simplex contained in  $D \cap \mathcal{F}_E$ . Then there exists a spherical subset  $V$  and cosets  $wW_V = w'W_V$  where  $w$  is odd and  $w'$  is even. Let  $v = w^{-1}w'$ . Since  $w$  is odd and  $w'$  is even,  $v$  must contain an odd number of  $t$ 's in any of its reduced expressions. Therefore  $t \in V$  and  $\sigma \in \partial_{in}(D)$ .  $\square$

As before, let  $\mathcal{F}_E$  denote the union of all even colors, and now let  $\mathcal{F}_O$  denote the union of a sub-collection of the odd colors. Let  $\mathcal{F}_{E'} = \mathcal{F}_E \cup \mathcal{F}_O$  and let  $F_o$  be an odd color not in  $\mathcal{F}_O$ . Then by 3.13,

$$F_o \cap \mathcal{F}_{E'} = (F_o \cap \mathcal{F}_E) \cup (F_o \cap \mathcal{F}_O) = \partial_{in}(F_o) \cup (F_o \cap \mathcal{F}_O).$$

Any 0-simplex in  $F_o$  which is also in a different color must be of the form  $wW_V$ , where  $w$  is a vertex of  $F_o$  and  $V \in \mathcal{S}_{\geq T}$ . Therefore  $(F_o \cap \mathcal{F}_O) \subset \partial_{in}(F_o)$  and  $F_o \cap \mathcal{F}_{E'} = \partial_{in}(F_o)$ .

It is clear from the product structure on boundary collars that  $\partial_{in}(F_o) \cong F_o \cap \partial\Omega$ , the latter a disjoint collection of components of  $\partial\Omega$ . Since  $L$  is flag, we have a 1-1 correspondence between cells of any component of  $\partial\Omega$  and cells of  $\Sigma(W_{U-t}, U-t)_{cc}$ . Denote by  $L_t$  the link in  $L$  of the vertex corresponding to  $t$ , it is a triangulation of  $\mathbb{S}^2$  and it is isomorphic to the nerve of  $(W_{U-t}, U-t)$ . Then since Conjecture 1.2 is true in dimension 3,

$$\mathcal{H}_i(F_o \cap \mathcal{F}_{E'}) = 0, \tag{3.3}$$

for all  $i$ .

**Proposition 3.14.** *Let  $(W, S)$  be an even Coxeter system whose nerve,  $L$  is a flag triangulation of  $\mathbb{S}^3$ . Let  $t \in S$ . Then  $\mathcal{H}_*(\Omega(S, t), \partial\Omega(S, t)) = 0$  for  $* = 3, 4$ .*

*Proof.* We first show that  $\mathcal{H}_4(\Omega, \partial\Omega) = 0$ . Consider the long exact sequence of the pair  $(\Omega, \partial\Omega)$ :

$$\rightarrow \mathcal{H}_4(\Omega) \rightarrow \mathcal{H}_4(\Omega, \partial\Omega) \rightarrow \mathcal{H}_3(\partial\Omega) \rightarrow$$

$\Omega$  is a 4-dimensional manifold with boundary, so  $\mathcal{H}_4(\Omega) = 0$  and  $\mathcal{H}_3(\partial\Omega) = 0$ . So by exactness,  $\mathcal{H}_4(\Omega, \partial\Omega) = 0$ .

Let  $\mathcal{F}_{E'}$  denote the union of a collection of even colors or the union of all evens and a collection of odd colors. Let  $F$  be a color not contained in  $\mathcal{F}_{E'}$  (if  $\mathcal{F}_{E'}$  is not all the even colors, require that  $F$  be an even color). Let  $\partial_{E'} = \mathcal{F}_{E'} \cap \partial\Omega$  and let  $\partial_F = F \cap \partial\Omega$ . Note that  $\partial_{E'} \cap \partial_F = \emptyset$  and consider the relative Mayer-Vietoris sequence of the pair  $(\mathcal{F}_{E'} \cup F, \partial_{E'} \cup \partial_F)$ :

$$\dots \rightarrow \mathcal{H}_3(\mathcal{F}_{E'}, \partial_{E'}) \oplus \mathcal{H}_3(F, \partial_F) \rightarrow \mathcal{H}_3(\mathcal{F}_{E'} \cup F, \partial_{E'} \cup \partial_F) \rightarrow \mathcal{H}_2(\mathcal{F}_{E'} \cap F) \rightarrow \dots$$

Assume that  $\mathcal{H}_3(\mathcal{F}_{E'}, \partial_{E'}) = 0$ . Each color retracts onto its boundary, so  $\mathcal{H}_3(F, \partial_F) = 0$ . If  $F$  is even, then the last term vanishes by 3.12, if  $F$  is odd, then the last term vanishes by (3.3). In either case, exactness implies that  $\mathcal{H}_3(\mathcal{F}_{E'} \cup F, \partial_{E'} \cup \partial_F) = 0$ . It follows from induction that  $\mathcal{H}_3(\Omega, \partial\Omega) = 0$ .  $\square$

## 4 The $\ell^2$ -homology of $\Sigma$

**Lemma 4.1.** *Let  $V \subseteq S$  and let  $T \subseteq V$  be a spherical subset with  $\text{Card}(T) = 2$ . Then  $\mathcal{H}_4(\Omega(V, T), \partial\Omega(V, T)) = 0$ .*

*Proof.* If  $\mathcal{S}(V)_{>T}^{(4)} = \emptyset$ , then  $\Omega(V, T)$  does not contain 4-dimensional cells, and we are done. So assume that  $\mathcal{S}(V)_{>T}^{(4)} \neq \emptyset$ . The codimension 1 faces of 4-cells of  $\Omega(V, T)$  are either faces of one other 4-cell in  $\Omega(V, T)$  ( $\Sigma$  is a 4-manifold), or they are free faces, i.e they are not faces of any other 4-cell in  $\Omega(V, T)$ .

Suppose that cells of type  $T' \in \mathcal{S}(V)_{>T}^{(4)}$  have a co-dimension one face of type  $F$  which is a face of another 4-cell in  $\Omega(V, T)$  of type  $T''$ . Then any relative 4-cycle must be constant on adjacent cells of type  $T'$  and  $T''$ , where  $T' = R \cup \{r\}$ , and  $T'' = R \cup \{s\}$ , for some  $R \in \mathcal{S}(V)^{(3)}$  and  $r, s \in V$ . Since  $L$  is flag and 3-dimensional,  $m_{rs} = \infty$ . So in this case, there is a sequence of adjacent 4-cells with vertex sets  $W_{T'}, W_{T''}, sW_{T'}, srW_{T''}, srsW_{T'}, sr srW_{T''}, \dots$ . Hence, this constant must be 0.

Now suppose that for a given 4-cell of  $\Omega(V, T)$ , every co-dimension one face is free. This cell has faces not contained in  $\partial\Omega(V, T)$ , so relative 4-cycles cannot be supported on this cell.  $\square$

Let  $V \subseteq S$ , be arbitrary;  $T \subseteq V$  spherical,  $\Omega := \Omega(V, T)$ ,  $\partial\Omega := \partial\Omega(V, T)$ . Recall that  $\Sigma(V)$  is the subcomplex of  $\Sigma_{cc}$  consisting of cells of type  $T'$ , with  $T' \subseteq V$ . We have excision isomorphisms from [5]:

$$C_*(\Omega(V, T), \partial\Omega) \cong C_*(\Sigma(V), \widehat{\Omega}(V, T)), \quad (4.1)$$

and for any  $s \in T$  and  $T' := T - s$ ,

$$C_*(\Sigma(V - s), \widehat{\Omega}(V - s, T')) \cong C_*(\widehat{\Omega}(V, T), \widehat{\Omega}(V, T')). \quad (4.2)$$

Set  $\widehat{\Omega} := \widehat{\Omega}(V, T)$ , and  $\widehat{\Omega}' := \widehat{\Omega}(V, T')$ . Consider the long, weakly exact sequence of the triple  $(\Sigma(V), \widehat{\Omega}, \widehat{\Omega}')$ :

$$\dots \rightarrow \mathcal{H}_*(\widehat{\Omega}, \widehat{\Omega}') \rightarrow \mathcal{H}_*(\Sigma(V), \widehat{\Omega}') \rightarrow \mathcal{H}_*(\Sigma(V), \widehat{\Omega}) \rightarrow \dots$$

By (4.1) and (4.2), the left hand term excises to the homology of the  $(V-s, T')$ -ruin, the right hand term to that of the  $(V, T)$ -ruin and the middle term to that of the  $(V, T')$ -ruin; leaving the sequence:

$$\dots \rightarrow \mathcal{H}_*(\Omega(V-s, T'), \partial) \rightarrow \mathcal{H}_*(\Omega(V, T'), \partial) \rightarrow \mathcal{H}_*(\Omega(V, T), \partial) \rightarrow \dots \quad (4.3)$$

**Proposition 4.2.** *Let  $(W, S)$  be an even Coxeter system, whose nerve  $L$  is a flag triangulation of  $\mathbb{S}^3$ . Let  $V \subseteq S$  and  $t \in V$ . Then*

$$\mathcal{H}_*(\Omega(V, t), \partial\Omega(V, t)) = 0, \quad (4.4)$$

for  $*$  = 3, 4.

*Proof.* It is clear that  $\mathcal{H}_*(\Omega(V, t)) = 0$  for  $*$  = 3, 4 whenever  $\text{Card}(V) \leq 2$ , so we may assume that  $\text{Card}(V) > 2$ . We show (4.4) by induction on  $\text{Card}(S-V)$ , Proposition 3.14 giving us the base case. Let  $V = V' \cup s$  and  $t \in V'$ . Assume (4.4) holds for  $V$ . If  $m_{st} = \infty$  then  $\Omega(V', t) = \Omega(V, t)$  and we are done. Otherwise, consider the sequence in (4.3), taking  $T = \{s, t\}$ ,  $T' = \{t\}$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{H}_4(\Omega(V', t), \partial) & \rightarrow & \mathcal{H}_4(\Omega(V, t), \partial) & \rightarrow & \mathcal{H}_4(\Omega(V, \{s, t\}), \partial) \rightarrow \\ & & \rightarrow & \mathcal{H}_3(\Omega(V', t), \partial) & \rightarrow & \mathcal{H}_3(\Omega(V, t), \partial) & \rightarrow \dots \end{array}$$

$\mathcal{H}_*(\Omega(V, t), \partial) = 0$  for  $*$  = 3, 4 by assumption and  $\mathcal{H}_4(\Omega(V, \{s, t\}), \partial) = 0$  by 4.1. So by exactness,  $\mathcal{H}_4(\Omega(V', t), \partial) = 0$ .  $\square$

**The Main Theorem 4.3.** *Let  $(W, S)$  be an even Coxeter system whose nerve  $L$  is a flag triangulation of  $\mathbb{S}^3$  and let  $\Sigma = \Sigma(W, S)$ . Then*

$$\mathcal{H}_*(\Sigma) = 0 \text{ for } * \neq 2.$$

*Proof.* Let  $V \subseteq S$  and  $t \in V$ . Consider the following form of (4.3), where  $T = \{t\}$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{H}_4(\Sigma(V-t)) & \rightarrow & \mathcal{H}_4(\Sigma(V)) & \rightarrow & \mathcal{H}_4(\Omega(V, t), \partial) \rightarrow \\ & & \rightarrow & \mathcal{H}_3(\Sigma(V-t)) & \rightarrow & \mathcal{H}_3(\Sigma(V)) & \rightarrow \mathcal{H}_3(\Omega(V, t), \partial) \rightarrow \dots \end{array}$$

By Proposition 4.2,  $\mathcal{H}_*(\Omega(V, t), \partial) = 0$  for  $*$  = 3, 4. So by exactness,

$$\mathcal{H}_*(\Sigma(V-t)) \cong \mathcal{H}_*(\Sigma(V)),$$

for  $*$  = 3, 4. It follows that  $\mathcal{H}_*(\Sigma) \cong \mathcal{H}_*(\Sigma(\emptyset)) = 0$  for  $*$  = 3, 4 and hence, by Poincaré duality,  $\mathcal{H}_*(\Sigma) = 0$  for  $*$   $\neq$  2.  $\square$

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